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CHEBYSHEV FITS MADE TO EQUALLY-SPACED DATA

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ABSTRACT

This memorandum presents a method of fitting equally-spaced data by the use of Chebyshev polynomials. The method is explained and tables are given to facilitate its application. The tables provide for fitting with all ordinates adjusted, fitting with zero error at one end point, or fitting with zero error at both end points as the situation may demand.

INTRODUCTION

The advantages of using Chebyshev polynomials in the polynomial approximation of a function are well known (see reference 1). A description of the simple method of using ordinary differences in the fitting of equally-spaced data seems to have been omitted by most writers. This memorandum makes an attempt to supply such a description.

PROPERTIES OF CHEBYSHEV POLYNOMIALS USED IN FITTING EQUALLY-SPACED DATA

The Chebyshev polynomial $T_n(u) = \cos(n \arccos u)$ has the well known property that for $-1 \leq u \leq 1$,

$$-1 \leq T_n(u) \leq 1 ,$$

with every local maximum value being 1 and every local minimum value being -1.

A polynomial of degree n, $P_n(x)$, can be approximated in the interval $a \leq x \leq b$ by a polynomial of degree $n - 1$ by using

$$P_n(x) = P_{n-1}(x) + K_n T_n(u) , \tag{1}$$

where u and x are related by a linear transformation. The maximum magnitude of error committed by using P_{n-1} instead of P_n will be $|K_n|$ provided u lies within the interval $-1 \leq u \leq 1$ for $a \leq x \leq b$.

Differences or derivatives can be used in finding K_n and P_{n-1}. The advantage of using differences with equally-spaced data is that the transformation relating u and x enters only in identifying the adjustments made to the ordinates of $y = P_n(x)$ to give points which lie on the curve $y = P_{n-1}(x)$. Thus the needs for "shifted" Chebyshev polynomials and for "normalizing" the argument of the function being studied do not arise. It will be assumed that Newton divided difference, Lagrange, or graphical methods have been used to reduce any unequally-spaced data and that the data are therefore available for equally-spaced arguments.

With data available for $x = a$, $x = a + h$, $x = a + 2h$, . . . $x = a + nh = b$, the values of $K_n T_n(u)$ can be computed by finding the nth difference of the function and multiplying by the correction factors from the appropriate table. Subtracting values of $K_n T_n(u)$ from values of the function will give ordinates for a polynomial curve of degree $n - 1$. For finding a lower degree approximation, a set of n equally-spaced points on $y = P_{n-1}(x)$ can be obtained by interpolation and the process repeated using the n-point table. The number of steps thus taken will be decided from the degree of the polynomial approximation required and the behavior of the successive values of K_n.

TABLES OF T_n USED IN FITTING WITH ALL ORDINATES ADJUSTED

For this case $u = -1$ at $x = a$ and $u = 1$ at $x = b$.

TABLE 1. FIRST DEGREE FIT (THREE POINTS)

$$\Delta^2 y = y_2 - 2y_1 + y_0 ,$$

$$T_2(u) = 2u^2 - 1 .$$

u	$T(u)$
-1	1
0	-1
1	1

$$\Delta^2 T_2 = 4 ,$$

$$K_2 = 0.25 \ \Delta^2 P_2(x) .$$

TABLE 2. SECOND DEGREE FIT (FOUR POINTS)

$$\Delta^3 y = y_3 - 3y_2 + 3y_1 - y_0 ,$$

$$T_3(u) = 4u^3 - 3u .$$

$3u$	$27\ T_3(u)$
-3	-27
-1	23
1	-23
3	27

$$9\ \Delta^3 T_3 = 64 ,$$

$$K_3 = 0.140\ 625\ \Delta^3 P_3(x) .$$

TABLE 3. THIRD DEGREE FIT (FIVE POINTS)

$$\Delta^4 y = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 \ ,$$

$$T_4(u) = 8u^4 - 8u^2 + 1$$

$2u$	$2\,T_4(u)$
-2	2
-1	-1
0	2
1	-1
2	2

$$\Delta^4 T_4 = 12 \ ,$$

$$K_4 = 0.08333 \ . \ . \ . \ \Delta^4 P_4(x) \ .$$

TABLE 4. FOURTH DEGREE FIT (SIX POINTS)

$$\Delta^5 y = y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 \ ,$$

$$T_5(u) = 16u^5 - 20u^3 + 5u$$

u	$T_5(u)$
-1.0	-1.00000
-0.6	0.07584
-0.2	-0.84512
0.2	0.84512
0.6	-0.07584
1.0	1.00000

$$\Delta^5 T_5 = 19.6608 \ ,$$

$$K_5 = 0.05086263 \ \Delta^5 P_5(x) \ .$$

TABLE 5. FIFTH DEGREE FIT (SEVEN POINTS)

$$\Delta^6 y = y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0 ,$$

$$T_6(u) = 32u^6 - 48y^4 + 18u^2 - 1 .$$

$3u$	$729\ T_6(u)$
-3	729
-2	239
-1	329
0	-729
1	329
2	239
3	729

$$81\ \Delta^6 T_6 = 2560 ,$$

$$K_6 = 0.031640625\ \Delta^6 P_6(x) .$$

TABLE 6. SIXTH DEGREE FIT (EIGHT POINTS)

$$\Delta^7 y = y_7 - 7y_6 + 21y_5 - 35y_4 + 35y_3 - 21y_2 + 7y_1 - y_0 ,$$

$$T_7(u) = 64u^7 - 112u^5 + 56u^3 - 7u .$$

$7u$	$823,543\ T_7(u)$
-7	-823,543
-5	-539,285
-3	33,933
-1	694,511
1	-694,511
3	- 33,933
5	539,285
7	823,543

$$823,543\ \Delta^7 T_7 = 41,287,680 ,$$

$$K_7 = 0.01994646\ \Delta^7 P_7(x) .$$

TABLE 7. SEVENTH DEGREE FIT (NINE POINTS)

$$\Delta^8 y = y_8 - 8y_7 + 28y_6 - 56y_5 + 70y_4 - 56y_3 + 28y_2 - 8y_1 + y_0 ,$$

$$T_8(u) = 128u^8 - 256u^6 + 160u^4 - 32u^2 + 1$$

u	$512\ T_8(u)$
-1.00	512
-0.75	449
-0.50	-256
-0.25	-223
0.00	512
0.25	-223
0.50	-256
0.75	449
1.00	512

$$512\ \Delta^8 T_8 = 40,320 ,$$

$$K_8 = 0.01269841\ \Delta^8 P_8(x) .$$

The method of extending these tables is obvious. Higher degree Chebyshev polynomials can be obtained from

$$T_{n+1}(u) = 2u T_n(u) - T_{n-1}(u) .$$

TABLES OF T_n USED IN FITTING WITH ZERO ERROR FOR $x = a$

For this case $u = u_-$ for $x = a$ and $u = 1$ for $x = b$, where u_- is the root of $T_n(u) = 0$ which is nearest $u = -1$. Values of $T_n(u)$ are then computed for equally-spaced arguments in the interval $u_- \le u \le 1$.

TABLE 2a. SECOND DEGREE FIT (FOUR POINTS)

Zero error for $x = a$

$$\Delta^3 y = y_3 - 3y_2 + 3y_1 - y_0 ,$$

$$T_3(u) = 4u^3 - 3u ,$$

$$u_- = - 0.86602\ 54038 .$$

u	$T_3(u)$
- 0.86602 54038	0.00000 00000
- 0.24401 69359	0.67393 15713
0.37799 15321	- 0.91794 85072
1.00000 00000	1.00000 00000

$$\Delta^3 T_3 = 5.77564\ 02355 ,$$

$$K_3 = 0.17314\ 0978\ \Delta^3 P_3(x) .$$

TABLE 3a. THIRD DEGREE FIT (FIVE POINTS)

Zero error for $x = a$

$$\Delta^4 y = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 ,$$

$$T_4(u) = 8u^4 - 8u^2 + 1 ,$$

$$u_- = -0.92387\ 95325 .$$

u	$T_4(u)$
- 0.92387 95325	0.00000 00000
- 0.44290 96494	- 0.26149 35811
0.03806 02338	0.98842 81359
0.51903 01169	- 0.57456 05703
1.00000 00000	1.00000 00000

$$\Delta^4 T_4 = 10.27478\ 54210 ,$$

$$K_4 = 0.09732\ 563349\ \Delta^4 P_4(x) .$$

TABLE 4 a. FOURTH DEGREE FIT (SIX POINTS)

Zero error for $x = a$

$$\Delta^5 y = y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0$$

$$T_5(u) = 16u^5 - 20u^3 + 5u \; ,$$

$$u_- = -\ 0.95105\ 65163$$

u	$T_5(u)$
- 0.95105 65163	0.00000 00000
- 0.56084 52130	- 0.16381 92820
- 0.17063 39097	- 0.75512 06893
0.21957 73934	0.89431 87770
0.60978 86967	- 0.13694 54280
1.00000 00000	1.00000 00000

$$\Delta^5 T_5 = 17.3700\ 2549$$

$$K_5 = 0.05757\ 04394\ \Delta^5 P_5(x)$$

TABLE 5a. FIFTH DEGREE FIT (SEVEN POINTS)

Zero error for $x = a$

$$\Delta^6 y = y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6\,y_1 + y_0 \ ,$$

$$T_6(u) = 32u^6 - 48u^4 + 18u^2 - 1 \ ,$$

$$u_- = -\,0.96592\ 58263 \ .$$

u	$T_6(u)$
- 0.96592 58263	0.00000 00000
- 0.63827 15219	0.53024 02655
- 0.31061 72175	0.31860 50447
0.01703 70870	- 0.99477 93213
0.34469 13912	0.51470 51628
0.67234 56956	0.28417 94180
1.00000 00000	1.00000 00000

$$\Delta^6 T_6 = 28.5087\ 2144 \ ,$$

$$K_6 = 0.03507\ 69825\ \Delta^6 P_6(x) \ .$$

TABLE 6 a. SIXTH DEGREE FIT (EIGHT POINTS)

Zero error for $x = a$

$$\Delta^7 y = y_7 - 7y_6 + 21y_5 - 35y_4 + 35y_3 - 21y_2 + 7y_1 - y_0 ,$$

$$T_7(u) = 64u^7 - 112u^5 + 56u^3 - 7u ,$$

$$u_- = -0.97492\ 79100 .$$

u	$T_7(u)$
- 0.97492 79100	0.0000 0000
- 0.69279 53514	- 0.7990 2858
- 0.41066 27928	0.1783 6705
- 0.12853 02342	0.7846 9717
0.15360 23242	- 0.8817 1724
0.43573 48828	0.0143 8894
0.71786 74414	0.6272 7220
1.00000 00000	1.0000 0000

$$\Delta^7 T_7 = 45.896859 ,$$

$$K_7 = 0.02178\ 7984\ \Delta^7 P_7(x) .$$

TABLES OF T_n USED IN FITTING WITH ZERO ERROR FOR $x = a$ AND FOR $x = b$

For this case $u = u_-$ for $x = a$ and $u = u_+$ for $x = b$, where u_- and u_+ are the roots of $T_n(u) = 0$ nearest $u = -1$ and $u = 1$, respectively. Values of $T_n(u)$ are then computed for equally-spaced arguments in the interval $u_- \le u \le u_+$.

TABLE 3b. THIRD DEGREE FIT (FIVE POINTS)

Zero error for $x = a$ and for $x = b$

$$\Delta^4 y = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 ,$$

$$T_4(u) = 8u^4 - 8u^2 + 1 ,$$

$$u_- = - u_+ = - 0.92387\ 95325 .$$

u	$T_4(u)$
- 0.92387 95325	0.00000 00000
- 0.46193 97662	- 0.34283 00857
0.00000 00000	1.00000 00000
0.46193 97662	- 0.34283 00857
0.92387 95325	0.00000 00000

$$\Delta^4 T_4 = 8.74264\ 06856 ,$$

$$K_4 = 0.11438\ 19169\ \Delta^4 P_4(x) .$$

TABLE 4b. FOURTH DEGREE FIT (SIX POINTS)

Zero error for $x = a$ and for $x = b$

$$\Delta^5 y = y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 ,$$

$$T_5(u) = 16u^5 - 20u^3 + 5u ,$$

$$u_- = - u_+ = - 0.95105\ 65163 .$$

u	$T_5(u)$
- 0.95105 65163	0.00000 00000
- 0.57063 39098	- 0.10501 08166
- 0.19021 13033	- 0.81740 21616
0.19021 13033	0.81740 21616
0.57063 39098	0.10501 08166
0.95105 65163	0.00000 00000

$$\Delta^5 T_5 = 15.297\ 9351 ,$$

$$K_5 = 0.06536\ 82995\ \Delta^5 P_5(x) .$$

TABLE 5b. FIFTH DEGREE FIT (SEVEN POINTS)

Zero error for $x = a$ and for $x = b$

$$\Delta^6 y = y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0 ,$$

$$T_6(u) = 32u^6 - 48u^4 + 18u^2 - 1 ,$$

$$u_- = - u_+ = - 0.96592\ 58263 .$$

u	$T_6(u)$
- 0.96592 58263	0.00000 00000
- 0.64395 05508	0.49208 59669
- 0.32197 52754	0.38581 81249
0.00000 00000	- 1.00000 00000
0.32197 52754	0.38581 81249
0.64395 05508	0.49208 59669
0.96592 58263	0.00000 00000

$$\Delta^6 T_6 = 25.6695\ 1214 ,$$

$$K_6 = 0.03895\ 6[illegible]201\ \Delta^6 P_6(x) .$$

TABLE 6b. SIXTH DEGREE FIT (EIGHT POINTS)

Zero error for $x = a$ and for $x = b$

$$\Delta^7 y = y_7 - 7y_6 + 21y_5 - 35y_4 + 35y_3 - 21y_2 + 7y_1 - y_0 ,$$

$$T_7(u) = 64u^7 - 112u^5 + 56u^3 - 7u ,$$

$$u_- = - u_+ = - 0.97492\ 79100$$

u	$T_7(u)$
- 0.97492 79100	0.0000 0000
- 0.69537 70786	- 0.7836 8357
- 0.41582 62472	0.1391 8140
- 0.13627 54158	0.8174 1285
0.13627 54158	- 0.8174 1285
0.41582 62472	- 0.1391 8140
0.69537 70786	0.7836 8357
0.97492 79100	0.0000 0000

$$\Delta^7 T_7 = 40.4017\ 1072 ,$$

$$K_7 = 0.02475\ 1427\ \Delta^7 P_7(x) .$$

TABLES OF CORRECTION FACTORS USED IN FITTING

Using equation (1),

$$P_n(x) = P_{n-1}(x) + K_n T_n(u) \ , \tag{1}$$

it is seen that

$$\Delta^n P_n = K_n \ \Delta^n T_n \ ,$$

from which

$$K_n = (\Delta^n T_n)^{-1} \ \Delta^n P_n \ .$$

Having values of T_n and $\Delta^n T_n$ from the preceding tables,

$$[(\Delta^n T_n)^{-1} \ T_n(u)] = C_n$$

can be computed. This correction factor, C_n, would then be multiplied by $\Delta^n P$ to give the amount which should be subtracted from ordinates on the $y = P_n(x)$ curve to obtain points on the $y = P_{n-1}(x)$ curve.

TABLE 8. CORRECTION FACTORS FOR FIRST DEGREE FIT (THREE POINTS)

x	C_2
a	0.250
$a + h$	- 0.250
$a + 2h$	0.250
x_m	0.250

$$P_1(x_i) = P_2(x_i) - \Delta^2 P_2 C_2(x_i) \ ,$$

$$|K_2| = \text{Maximum } |P_2 - P_1| = |\Delta^2 P_2| \ C_2(x_m) \ .$$

TABLE 9. CORRECTION FACTORS FOR SECOND DEGREE FIT (FOUR POINTS)

	C_3	C'_3
a	- 0.1406 2500	0.0000 0000
$a + h$	0.1197 9167	0.1166 8517
$a + 2h$	- 0.1197 9167	- 0.1589 3450
$a + 3h$	0.1406 2500	0.1731 4098
x_m	0.1406 2500	0.1731 4098

$$P_2(x_i) = P_3(x_i) - \Delta^3 P_3 \left\{ \quad \right\}_{x_i} ,$$

$$|E_3| = \text{Maximum } |P_3 - P_2| = |\Delta^3 P_3| \left\{ \quad \right\}_{x_m} ,$$

$$\left\{ \quad \right\} \begin{array}{l} = C_3 \text{ (all ordinates adjusted)} , \\ = C'_3 \text{ (zero error for } x = a) . \end{array}$$

TABLE 10. CORRECTION FACTORS FOR THIRD DEGREE FIT (FIVE POINTS)

x	C_4	C_4'	C_4''
a	0.0833 3333	0.0000 0000	0.0000 0000
$a + h$	- 0.0416 6667	- 0.0254 5003	- 0.0392 1356
$a + 2h$	0.0833 3333	0.0961 9939	0.1143 8192
$a + 3h$	- 0.0416 6667	- 0.0559 1947	- 0.0392 1356
$a + 4h = b$	0.0833 3333	0.0973 2563	0.0000 0000
x_m	0.0833 3333	0.0973 2563	0.1143 8192

$$P_3(x_i) = P_4(x_i) - \Delta^4 P_4 \{\quad\}_{x_i} ,$$

$$|K_4| = \text{Maximum } |P_4 - P_3| = |\Delta^4 P_4| \{\quad\}_{x_m} ,$$

$$\{\quad\} \begin{cases} = C_4 \text{ (all ordinates adjusted)} , \\ = C_4' \text{ (zero error for } x = a) , \\ = C_4'' \text{ (zero error for } x = a \text{ and for } x = b) . \end{cases}$$

TABLE 11. CORRECTION FACTORS FOR FOURTH DEGREE FIT (SIX POINTS)

x	C_5	C_5'	C_5''
a	- 0.0508 6263	0.0000 0000	0.0000 0000
$a + h$	0.0038 5742	- 0.0094 3115	- 0.0068 6438
$a + 2h$	- 0.0429 8503	- 0.0435 3020	- 0.0534 3219
$a + 3h$	0.0429 8503	0.0514 8632	0.0534 3219
$a + 4h$	- 0.0038 5742	- 0.0078 8401	0.0066 5438
$a + 5h = b$	0.0508 6263	0.0575 7044	0.0000 0000
x_m	0.0508 6263	0.0575 7044	0.0653 6830

$$P_4(x_i) = P_5(x_i) - \Delta^5 P_5 \{\quad\}_{x_i} ,$$

$$|K_5| = \text{Maximum } |P_5 - P_4| = |\Delta^5 P_5| \{\quad\}_{x_m} ,$$

$$\left\{\quad\right\} \begin{cases} = C_5 \text{ (all ordinates adjusted) ,} \\ = C_5' \text{ (zero error for } x = a) , \\ = C_5'' \text{ (zero error for } x = a \text{ and for } x = b) . \end{cases}$$

TABLE 12. CORRECTION FACTORS FOR FIFTH DEGREE FIT (SEVEN POINTS)

x	C_e	C_e'	C_e''
a	0.0316 40625	0.0000 00000	0.0000 00000
$a + h$	0.0103 73264	0.0185 99230	0.0191 70055
$a + 2h$	0.0142 79514	0.0111 75704	0.0150 30209
$a + 3h$	- 0.0316 40625	- 0.0348 93860	- 0.0389 56720
$a + 4h$	0.0142 79514	0.0180 54305	0.0150 30209
$a + 5h$	0.0103 73264	0.0099 68157	0.0191 70055
$a + 6h = b$	0.0316 40625	0.0350 76985	0.0000 00000
x_m	0.0316 40625	0.0350 76985	0.0389 56720

$$P_s(x_i) = P_e(x_i) - \Delta^e P_e \{ \quad \}_{x_i} ,$$

$$|E_e| = \text{Maximum } |P_e - P_s| = |\Delta^e P_e| \{ \quad \}_{x_m} ,$$

$$\{ \quad \} = C_e \text{ (all ordinates adjusted)} ,$$
$$= C_e' \text{ (zero error for } x = a) ,$$
$$= C_e'' \text{ (zero error for } x = a \text{ and } x = b) .$$

TABLE 13. CORRECTION FACTORS FOR SIXTH DEGREE FIT (EIGHT POINTS)

x	C_7	C_7'	C_7''
a	- 0.0199 46459	0.0000 00000	0.0000 00000
$a + h$	- 0.0130 61645	- 0.0174 09221	- 0.0193 97287
$a + 2h$	0.0008 21867	0.0038 86258	0.0034 44938
$a + 3h$	0.0168 21265	0.0170 96969	0.0202 32135
$a + 4h$	- 0.0168 21265	- 0.0192 10841	- 0.0202 32135
$a + 5h$	- 0.0008 21867	0.0003 13506	- 0.0034 44938
$a + 6h$	0.0130 61645	0.0136 66996	0.0193 97287
$a + 7h = b$	0.0199 46459	0.0217 87983	0.0000 00000
x_m	0.0199 46459	0.0217 87983	0.0247 51427

$$P_8(x_i) = P_7(x_i) - \Delta^7 P_7 \{\quad\}_{x_i} \, ,$$

$$|E_7| = \text{Maximum } |P_7 - P_8| = |\Delta^7 P_7| \{\quad\}_{x_m} \, ,$$

$$\{\quad\} = \begin{cases} = C_7 \text{ (all ordinates adjusted)} \, , \\ = C_7' \text{ (zero error for } x = a) \, , \\ = C_7'' \text{ (zero error for } x = a \text{ and } x = b). \end{cases}$$

NUMERICAL EXAMPLE

In order to illustrate the proposed method, Tables 11 and 12 will be used to approximate

$$y = \ln(1 + x) \qquad , \qquad (0 \leq x \leq 1)$$

by a fourth degree polynomial. Suppose the following values are given.

$6x$	$\ln(1 + x)$
0	0.00000 0000
1	0.15415 0680
2	0.28768 2073
3	0.40546 5108
4	0.51082 5624
5	0.60613 5804
6	0.69314 7181

From the above values

$$\Delta^6 \ln(1 + x) = - 0.00025\,8428 \,.$$

Turning to Table 12 and remembering that $a = 0$ and $6h = 1$,

$$\Delta^6 P_6 \; C_6(a) = - 0.00000\,81768 \,,$$

$$\Delta^6 P_6 \; C_6(a + h) = - 0.00000\,26807 \,,$$

$$\Delta^6 P_6 \; C_6(a + 2h) = 0.00000\,36902, \text{ etc.}$$

These values are used to find points approximating the logarithmic curve and lying exactly on the curve $y = P_6(x)$.

$6x$	$P_6(x)$
0	0.00000 8177
1	0.15415 3361
2	0.28768 5763
3	0.40545 6931
4	0.51082 9314
5	0.60613 8484
6	0.69315 5358

These values can be used directly to obtain a fifth degree polynomial approximating $\ln(1 + x)$. To obtain a fourth degree approximation, fifth degree interpolation is used to obtain six points with equally spaced arguments.

x	$P_5(x)$
0	0.00000 8177
0.2	0.18232 8444
0.4	0.33646 9893
0.6	0.47000 1120
0.8	0.58779 2299
1.0	0.69315 5358

From these values

$$\Delta^5 P_5 = 0.00114\ 0176 \ .$$

Using Table 11, $a = 0$, $h = 0.2$

$$\Delta^5 P_5 \ C_5(a) = -\ 0.00005\ 7992 \ ,$$

$$\Delta^5 P_5 \ C_5(a + h) = \ 0.00000\ 4398 \ ,$$

$$\Delta^5 P_5 \ C_5(a + 2h) \quad -\ 0.00004\ 9010, \text{ etc.}$$

These values are used to find points approximating the logarithmic curve and lying exactly on the curve

$$y = P_4(x)$$

x	$P_4(x)$
0	0.00006 6170
0.2	0.18232 4046
0.4	0.33651 9903
0.6	0.46995 3109
0.8	0.58779 6597
1.0	0.69309 7365

The points on $y = P_4(x)$ give the coefficients in P_4 so that

$$\begin{aligned} \ln(1 + x) = P_4(x) = {} & 0.00006\ 6170 \\ & + 0.99627\ 6293\ x \\ & - 0.46644\ 2009\ x^2 \\ & + 0.21862\ 2302\ x^3 \\ & - 0.05542\ 5391\ x^4 \ , \end{aligned}$$

for

$$0 \le x \le 1 \ .$$

If the function being fitted had been a sixth degree polynomial, the maximum magnitude of error would not exceed $|K_6| + |K_5| \doteq 0.00006\ 6170$. It can be expected that in this case the maximum magnitude of error will not be significantly larger than this amount.

COMPARISON OF MAXIMUM MAGNITUDES OF ERROR FOR DIFFERENT TYPES OF FIT

The gain in accuracy resulting from the use of Chebyshev polynomials is shown in Table 14. The κ_n for unadjusted ordinates is the maximum magnitude of

$$\frac{s(s-1)(s-2)\ldots(s-n+1)}{1 \cdot 2 \cdot 3 \ldots n}$$

in the interval $0 \leq s \leq 1$.

In making an $(n-1)$st degree Chebyshev fit, the interval $a \leq x \leq b$ is divided into n equal parts instead of $n - 1$ equal parts as is the case for n point interpolation with unadjusted ordinates. Therefore

$$\kappa_n = \left(\frac{n-1}{n}\right)^n \{\quad\}_{x_m} ,$$

$$\{\quad\} \begin{cases} = C_n & \text{(all ordinates adjusted)} \\ = C_n' & (\text{zero error for } x = a) \\ = C_n'' & (\text{zero error for } x = a \text{ and } x = b) \ . \end{cases}$$

TABLE 14. $e_n = \kappa_n \Delta^n y$ = MAXIMUM MAGNITUDE OF ERROR IN n POINT $(x_0, x_0 + h, \ldots, x_0 + (n-1)h)$ INTERPOLATION

n	κ_n Unadjusted Ordinates	κ_n All Ordinates Adjusted	κ_n $y(a)$ Unadjusted; Other Ordinates Adjusted	κ_n $y(a)$ and $y(b)$ Unadjusted; Other Ordinates Adjusted
2	0.1250 0000	0.0625 0000	-	-
3	0.0641 5003	0.0416 6667	0.0513 0103	-
4	0.0416 6527	0.0263 6719	0.0307 94439	0.03619 11533
5	0.0302 6194	0.0166 6667	0.0188 6468	0.0214 1988
6	0.0234 7346	0.0105 9638	0.0117 4721	0.0130 4653
7	0.0190 1625	0.0067 8013	0.0074 0610	0.0084 1342
8	0.0158 8792	0.0043 6307		

APPLICATIONS OF METHOD

It is obvious that equation (1) can be used in many situations not discussed in this memorandum. Unequally-spaced data or data outside the interval $a \leq x \leq b$ could be used. It is hoped that the tables of correction factors will encourage the use of Chebyshev fits by those who have previously felt that the extra effort outweighed the benefits. These tables cover the cases most frequently encountered in making simple approximations at the desk. For machine computation, coding can be prepared for use of higher polynomials or other spacing of arguments.

REFERENCE

1. Lanczos, Cornelius, *Applied Analysis*, Prentice Hall, 1956